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The Wedderburn principal theorem and Shukla cohomology

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Abstract

The Wedderburn principal theorem states that a finite-dimensional algebra A over a perfect field F is a vector space direct sum of its radical ideal J and a subalgebra S: $A = S \oplus J$. The proof of this fact was deep for its time.

In a conceptual breakthrough, Hochschild found a cohomological proof of Wedderburn's theorem. This proof makes a reduction to the case where $J^2 = 0$. The quotient map $A \to A/J$ has a linear right inverse s. The $s(xy) - s(x)s(y)$ defines a J-valued 2-cocycle in Hochschild cohomology theory. Now A/J is a separable F-algebra, so has vanishing positive-dimensional cohomology groups; whence there exists a map $g: A/J \to J$ such that $s(xy) - s(x)s(y) = s(x)g(y) - g(xy) + g(x)s(y)$. Hence $\psi = s + g$ is a homomorphism of algebras that is a right inverse of $A \to A/J$. Taking S to be the subalgebra $\psi(A/J)$, $A = S \oplus J$ is satisfied.

If A is instead an algebra over a general commutative ring, a linear right inverse s might not exist: e.g., the natural surjection of $\mathbb{Z}$-algebras, $\mathbb{Z}_{p^2} \to \mathbb{Z}_p$, where p is prime. However, a set-theoretic right inverse t for $A \to A/J$ exists by the axiom of choice. Forming both $t(xy) - t(x)t(y)$ and $t(x + y) - t(x) - t(y)$, we show that these give a J-valued 2-cocycle in a more refined cohomology theory of algebras due to Shukla (1961). I give an updated account of the nuances of Shukla's cohomology theory, then obtain a fully generalized cohomological version of Wedderburn's theorem, and discuss its role in ring theory.

1. Introduction

About half a century after their appearance in mathematics, finite-dimensional algebras received a big impetus towards their classification through the work of J. H. M. Wedderburn. His work in the early 1900s is a constant source of powerful generalization – Artin algebras, Jacobson radical, Goldie rings, and Morita's theorems – and a source of specializations such as quiver algebras and division algebras. Wedderburn's main three theorems are (1) a simple finite-dimensional algebra is a full matrix algebra over a division algebra; (2) a semisimple finite-dimensional algebra is a direct product of simple algebras; and (3) a finite-dimensional algebra over a perfect field is

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a split extension of a semisimple subalgebra by a nilpotent ideal. The last theorem is especially celebrated as "the principal theorem" in many texts but has been at times more routinely referred to as his "factorization theorem". In specific terms, the principal theorem states that a finite-dimensional algebra A over a perfect field has a semisimple subalgebra S and nilpotent ideal J such that $A = S \oplus J$ (a direct sum of vector subspaces). By taking some small matrix examples it is clear that S may not be unique, but any other semisimple subalgebra S' such that $A = S' \oplus J$ is conjugate to S by an element $1 - x$ for some element x in J (a theorem of Malcev).

In 1945 Hochschild introduced in a pair of articles in the Annals of Mathematics a cohomology theory of algebras over fields, and gave a conceptual, pathbreaking proof of the principal theorem. The idea is basically to prove that the exact sequence $0 \to J \to A \to A/J \to 0$ of natural algebra homomorphisms is split by an algebra homomorphism. First one observes that A/J is a separable algebra, then Hochschild introduces the separability element and characterizes separable algebras as having vanishing cohomology groups $HH^n(A,M)$ for $n = 1,2,3, \ldots$ and any A-bimodule M. He then shows that $HH^2(A,M)$ is in one-to-one correspondence with isomorphism classes of exact sequences of algebras $0 \to M \to B \xrightarrow{\sigma} A \to 0$, where $M^2 = 0$ (a "square-zero extension of A by M"): the cobounded 2-cocycles corresponding to $B =$ the semidirect product $A \times M$ with the natural maps and other 2-cocycles giving a nontrivial "twist" to the multiplication of the semidirect product. If f is an M-valued 2-cocycle one defines multiplication on $A \times M$ by the well-known formula

$$(a,m)(b,n) = (ab, an + mb + f(a,b)).$$

Conversely, a square-zero extension of A by M with a linear section ϕ of σ defines a "factor set", i.e., a bilinear M-valued map measuring the failure of ϕ to be multiplicative, and associativity of the multiplication in B implies a cocycle condition on the factor set. For the separable algebra A/J and the special case where $J^2 = 0$ this mplies that the factor set is cobounded, since $HH^2(A,J) = 0$, so that a homomorphic splitting exists. The general case is disposed of by an induction argument on the index of nilpotency of J. The theorem Hochschild proved as a result can be stated easily for algebras over any commutative ring k as the following vast generalization of the principal theorem.

Theorem 1.1. *If a unital k-algebra A contains a nilpotent ideal J such that*

(1) *A/J is a projective k-module,*

(2) *$HH^2(A/J, M) = 0$ for every A/J-bimodule M,*

then there exists in A a subalgebra S such that $A = S \oplus J$ as k-modules.

This theorem touches upon current research. Algebras enjoying property (ii), i.e., having vanishing second cohomology groups, are the so-called quasi-free algebras of

D. Quillen and J. Cuntz, are being revived because of their role in noncommutative differential geometry. Via the separability element, separable algebras generalize to separable extensions of algebras, and the subclass of separable extensions that also satisfy a condition of splitness possesses the desirable features of the basic construction of V. F. R. Jones, while unifying the disparate examples of finite index subgroups, coprime degree separable field extensions, matrix extensions, and type II_1 subfactors of finite index.

In the Hochschild theory it is important that every exact sequence be split over the ground ring k. Then every square-zero extension will have a linear splitting as demanded by the theory. But consider the general extension problem. For example, given $\mathbb{Z}_p$, the integers modulo a prime p, it is easy to compute $HH^2(\mathbb{Z}_p, \mathbb{Z}_p) = 0$, where $\mathbb{Z}_p$ is viewed as a $\mathbb{Z}$-algebra with its natural bimodule $\mathbb{Z}_p$. However, the square-zero extensions $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$ (one for each $p-1$ nonisomorphic inclusions followed by the canonical projection) are all nonisomorphic to the semidirect product. To account for this phenomenon one needs a general cohomology theory of algebras that takes the value $\mathbb{Z}_p$ on the algebra-bimodule pair $(\mathbb{Z}_p, \mathbb{Z}_p)$, and reduces to Hochschild's cohomology groups for algebras over fields. Such a cohomology theory was produced by Shukla in his thesis of 1961 [10], after crucial preparatory work by Mac Lane [6, 7].

In this article I would like to show how to generalize Hochschild's version of the principal theorem using Shukla cohomology. What we prove is Theorem 3.1, which is just the theorem above with Shukla's cohomology group replacing Hochschild's in condition (ii), and condition (i) dropped. This involves showing that two factor sets (one measuring failure of linearity, the other, multiplicativity, of a set-theoretic splitting function) form a Shukla 2-cocycle: a point with technical subtleties that seems not to have been verified in print. Shukla cohomology has led a shadowy existence since its inception, perhaps because of its computational complexity and lack of application.[1] We hope to make some small improvement to this situation by making a new computation (Eq. (1)) of Shukla cohomology in Section 2, after giving an updated account of the theory. We then state and prove our main theorem in some detail, also with the aim of making Shukla's theory digestible to interested readers. In Section 4 we conclude with a discussion of possible nice formulations of Wedderburn's principal theorem within general ring theory, armed with Shukla cohomology and information gained by Cegarra and Garzon [3] on the second Shukla group of torsion-free rings. We also point out that the well-known decomposition of divisible groups is an additive analog of the principal theorem, which leads to more success in the classification problem than in the corresponding problem for algebras.

[1] Recently, T. Pirashvili and F. Waldhausen have found a close connection between M. Bökstedt's topological Hochschild homology and Mac Lane homology, which is just a specialization of Shukla homology to the integers as ground ring.

2. Shukla cohomology

We first present Shukla's cohomology groups of an algebra A [10] via a standard complex, namely, the bar resolution of a certain differential graded algebra resolution of A.

Given a k-algebra A with unit, one can find a differential graded algebra (V, d) with augmentation $\varepsilon : (V, d) \to A$ (A equipped with zero differential and zero components in nonzero degree) (V, d) a free resolution of the k-module A,

$$\cdots \to V_n \xrightarrow{d_n} V_{n-1} \xrightarrow{d_{n-1}} \cdots V_0 \xrightarrow{\varepsilon} A \to 0.$$

Thus, V admits a graded product, d satisfies a Leibnitz rule, and ε is a homomorphism of k-algebras. First take V_0 to be the free k-module on basis $\{(a) \mid a \in A,\ a \neq 0\}$ and define V_n inductively as the free k-module on non-trivial elements of the $\ker(d_{n-1})$ (where $\varepsilon = d_0$). Then take $d_n : V_n \to V_{n-1}$ to be the linear extension of inclusion of basis elements. Similarly, we take $\varepsilon(\sum_{i=1}^n k_i(a_i)) = \sum_{i=1}^n k_i a_i$, where $k_i \in k$, $a_i \in A - 0$.

The cubical elements of Mac Lane [7] are embedded in this complex as follows. Denote by (a, b) the basis element of V_1 corresponding to $(a + b) - (a) - (b)$ in $\ker \varepsilon$, so that $d_1((a, b)) = (a + b) - (a) - (b)$. The other cubical elements also appear as special basis elements of the V_n's: e.g., the next level cubical element is the basis element

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

of V_2 mapping under d_2 to $(a + c, b + d) - (a, b) - (c, d) + (a, c) + (b, d) - (a + b, c + d)$ in $\ker d_1$.

The multiplication in V_0 is given on basis elements by $(a)(b) = (ab)$ and extended linearly. For basis elements $z \in V_n$ and $w \in V_m$ define the multiplication inductively by

$$zw = d_{n+m}^{-1}((d_n(z))w + (-1)^n z d_m(w)),$$

where $d_0 = 0$. For example $(a, b)(c) = (ac, bc)$. Note that V has a homogeneous k-base, i.e. its basis is closed under multiplication (as are the cubical elements). That V is a differential graded algebra augmentation of A and additively a free resolution of the k-module A is now clear.

Definition. V will be referred to as the *standard construction* over a k-algebra A. Let C_1 be the category of free resolutions of A that possess differential graded(d.g.) algebra structure with homogeneous k-base containing the unity element and with augmentation map to A that is a d.g. algebra epimorphism. V is easily seen to be the unique terminal object in C_1 [10].

Given a k-algebra A with A-bimodule M, the Shukla cohomology groups $HS^n(A, M)$ ($n = 0, 1, 2, \ldots$) are defined as the cohomology groups of the chain complex resulting from the application of the functor $\mathrm{Hom}(-, M)$ to the bar resolution of the differential graded algebra V [8] (M as a pullback V–bimodule via the augmentation).

Since we will need the nuts and bolts of this definition (at least for 2-cocycles), we present an interpretation of this definition as a bicomplex with resulting cohomology groups.

Taking V as before, the graded tensor product $V \otimes_k \cdots \otimes V = V^{\otimes r}$ (r times V) has a natural differential δ_d and grading $(V^{\otimes r})_s$ as an alternating sum of the differential d applied to each entry at a time, and grade gotten by adding the grades of homogeneous elements in a tensor element. Define cochain groups $C^{p,q} = \mathrm{Hom}_k((V^{\otimes p})_q, M)$ of the bicomplex $C^{**}(V \to A)$ with vertical differential $\delta_d : C^{p,q} \to C^{p,q+1}$ and horizontal differential $\delta_b : C^{p,q} \to C^{p+1,q}$ defined as follows (each u_i a homogeneous element of V of grade $|u_i|$):

$$\delta_d f(u_1, \dots, u_p) = -\sum_{i=1}^{p} (-1)^{e_{i-1}} f(u_1, \dots, du_i, \dots, u_p),$$

$$(f \in C^{p,q}, e_i = i + |u_1| + \cdots + |u_i|),$$

$$\delta_b f(u_1, \dots, u_{p+1}) = \varepsilon(u_1) f(u_2, \dots, u_{p+1}) + \sum_{i=1}^{p} (-1)^{e_i} f(u_1, \dots, u_i u_{i+1}, \dots, u_{p+1}) + (-1)^{q+p+1} f(u_1, \dots, u_p)\varepsilon(u_{p+1}).$$

We take $C^{0,0} = M$ and $\delta_b m(v_0) = \varepsilon(v_0)m - m\varepsilon(v_0)$. It may be computed that $\delta_b^2 = \delta_d^2 = 0 = \delta_d\delta_b + \delta_b\delta_d$.

Definition. The *Shukla cohomology* groups of A with values in M are defined to be the cohomology groups of the cochain complex ($C^n = \sum_{p+q=n} \oplus\, C^{p,q}$, $\delta = \delta_b + \delta_d$), i.e., the total complex of $C^{**}(A)$. We denote these cohomology groups by $HS^n(A,M)$; then

$$HS^n(A,M) = H^n(C^*, \delta).$$

We give a simple and important example. A Shukla 2-cocycle is a pair of multilinear M-valued functions $(f, g) \in \mathrm{Hom}_k(V_0 \otimes_k V_0, M) \oplus \mathrm{Hom}_k(V_1, M)$ satisfying the four conditions which we give only on cubical elements for the sake of simplicity:

1. $$\delta_b f((c),(d),(e)) = cf((d),(e)) - f((cd),(e)) + f((c),(de)) - f((c),(d))e = 0,$$

2. $$\delta_d g\left(\begin{pmatrix} c & d \\ e & f \end{pmatrix}\right) = g((c,e)) + g((d,f)) - g((c+d, e+f)) + g((c+e, d+f)) - g((c,d)) - g((e,f)) = 0,$$

3. $$\delta_d f((c,d),(e)) + \delta_b g((c,d),(e)) = f((c),(e)) + f((d),(e)) - f((c+d),((e)) + g((ce,de)) - g((c,d))e = 0$$

4. $$\delta_d f((c),(d,e)) + \delta_b g((c,d),(d,e)) = f((c),(d+e)) - f((c),(d)) - f((c),\ (e) + cg((d,e)) - g((cd,ce))e = 0$$

Note the resemblance of the horizontal differential δ_b to the coboundary b of the standard Hochschild cochain complex of A [4]. In fact $\varepsilon : V \to A$ induces a homomorphism of cochain complexes, and corresponding morphism of cohomologies, $\varepsilon^* : HH^n(A,M) \to HS^n(A,M)$. For example, a Hochschild 2-cochain $f(-,-)$ maps to $(f(\varepsilon(-),\varepsilon(-)),0)$. ε^* is an isomorphism of cohomologies in degree $n = 0$ and $n = 1$, and a monomorphism in $n = 2$. This follows from the spectral sequence of a bicomplex filtered by columns: when applied to $C^{*,*}(A)$, it gives $E_2^{n,0} = H_b^n H_d^0(C^{n,0}) = HH^n(A,M)$.

Shukla proves in [10] that there is a certain freedom of choice in choosing V when computing $HS^n(A,M)$ as the cohomology of the bar construction of V. Any object in the category C_1 of free resolutions over A possessing differential graded algebra structure with homogeneous k-base will suffice for V. It is also shown that it suffices to choose only a projective resolution V of A that possesses differential graded algebra structure and a multiplicative right inverse to the augmentation $\varepsilon : V \to A$ and a (set-theoretic) right inverse to each differential $d_n : V_n \to \ker d_{n-1}$: such resolutions over A form a category C_2.

For example, if A is projective over k, choose $V = A$, the trivial d.g. algebra with identity map augmentation, and computing $C^{**}(V \to A)$ we get all zeroes except a bottom row, so that $HS^n(A,M) = HH^n(A,M)$ for every $n \geq 0$.

Another example: if A is an algebra over a principal ideal domain (p.i.d.), we use the following simplified resolution V over A. $V_2 = V_3 = \cdots = 0$, V_0 and ε are as in the standard construction, and $V_1 = \ker \varepsilon$, which is automatically free over k. V_0 and V_1 have the same product as before except that $V_1^2 = 0$. Then V is an object in Shukla's category C_2, and $C^{**}(V \to A)$ is simplified since $C^{p,q} = 0$ if $q > p$.

A simplification in computing Shukla cohomology occurs by using normalized cochains; i.e., a Shukla cochain that vanishes if an argument is a scalar multiple of the unity in V_0. This is most easily seen by working with chains rather than cochains: a chain complex (D,δ) of Shukla chains with 1 appearing in some entry is easily shown to be contractible, so that the Shukla chain complex (C,δ) is homology equivalent to the quotient complex $(C/D,\delta)$ (cf. [8]).

As an application of normalized cochains we compute $HS^n(A,M)$ and $HH^n(A,M)$, where A is an algebra over a p.i.d. k with surjective unit map $u : k \to A$. Then take for V the resolution $\cdots \to 0 \to k \xrightarrow{d} k \xrightarrow{u} A \to 0$, where $d(n) = nr$ if $\ker(u) = (r)$. The normalized cochain bicomplex is simply

$$C^{p,q}(V \to A) = \begin{cases} 0, & p \neq q, \\ M, & p = q, \end{cases}$$

so that $HS^{2n}(A,M) = M$ and $HS^{2n+1}(A,M) = 0$ for each nonnegative integer n. The spectral sequence relating Hochschild and Shukla cohomologies readily implies that the Hochschild groups are zero except $HH^0(A,M) = M$.

For example,

$$HS^n(\mathbb{Z}_p,\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases} \tag{1}$$

The corresponding Hochschild groups are all zero. While the Hochschild group $HH^2(A, M)$ corresponds only to those square-zero extensions that are k-split, it is known that the elements of the group $HS^2(A, M)$ are in 1–1 correspondence with all the isomorphism classes of square-zero extensions of A by M. The argument for this fact is partially given in [10] by defining a multiplication and addition on $A \times M$ using a Shukla 2-cocycle; the other part, missing in [10], is provided in the proof of Theorem 3.1; i.e., we show that given a square-zero extension $0 \to M \to B \xrightarrow{p} A \to 0$ a factor set (f, g) (measuring deviation of a set-theoretic right inverse of p from preserving multiplication and addition) is a Shukla 2-cocycle. The full argument is then obtainable by a diligent checking of some details. In the introduction we spotted p nonisomorphic square-zero extensions of $\mathbb{Z}_p$ by its module $\mathbb{Z}_p$; whence Shukla cohomology is the right cohomology theory for the general extension problem mentioned in the introduction.

A further clarification in the relation of Hochschild and Shukla cohomologies was reached by Barr in the 1960s [1]. The guiding philosophy then was that adjoint functors give rise to cotriple cohomology theories that should include all the cohomology theories. Indeed, Shukla cohomology is up to a shift of $+1$ in dimension the cotriple cohomology of the adjoint functors F and G, where F sends the comma category of k-algebras over A to the underlying sets and their maps in *Set*, G is the free algebra functor on sets. One must apply the additive functor $\mathrm{Der}_k(-, M)$, derivations into an A-module M, to get cohomology with coefficients. Hochschild cohomology occurs the same way except that F and G pass to and from the category of vector spaces over k, F the functor to the underlying vector spaces, G the tensor algebra functor.

3. Main theorem

Theorem 3.1. *If A is a k-algebra with J a nilpotent ideal such that the quotient algebra A/J has vanishing second Shukla cohomology groups, i.e., $HS^2(A/J, M) = 0$ for every A/J-bimodule M,[2] then there exists a subalgebra S in A such that $A = S \oplus J$ as k-modules.*

Proof. First suppose that J is a nontrivial ideal satisfying $J^2 = 0$. Denoting A/J by B; i, the inclusion of J in A, and $p: A \to B$ the canonical projection homomorphism, we have the square-zero extension of B by J:

$$0 \longrightarrow J \xrightarrow{i} A \xrightarrow{p} B \longrightarrow 0.$$

[2] It suffices to suppose $HS^2(A/J, J^{2^i}/J^{2^{i+1}}) = 0$ for integers $i = 0, 1, \ldots, m$, where $2^m \le n < 2^{m+1}$ and n is the index of nilpotency of J. The bimodule structures are defined in a straightforward inductive way.

By the axiom of choice there exists a set-theoretic right inverse of p, call it $s: B \to A$ and choose it such that $s(0) = 0$ and $s(1) = 1$. One can make J into a B-bimodule by defining left and right actions simply as $bx = s(b)x$ and $xb = xs(b)$ (suppressing i): as a consequence of $J^2 = 0$, the actions are linear and associative, and indeed independent of choice of right inverse to p.

Let $\varepsilon: V \to B$ be the standard construction. Define factor sets $f: V_0 \times V_0 \to J$ and $g: V_1 \to J$ by the formulas

$$f((x),(y)) = s(xy) - s(x)s(y),$$

$$g\left(\left(\sum_{i=1}^{n} r_i(x_i)\right)\right) = \sum_{i=1}^{n} r_i s(x_i),$$

where $\sum_{i=1}^{n} r_i x_i = 0$ ($x, y, x_i \in B$ and $r_i \in k$). Thus, $\sum_{i=1}^{n} r_i(x_i) \in \ker \varepsilon$ whose preimage in the homogeneous k-base under $d_1: V_1 \to V_0$ is $(\sum_{i=1}^{n} r_i(x_i))$. By ordinary linear extension we define morphisms $f \in \mathrm{Hom}_k(V_0 \otimes_k V_0, J)$ and $g \in \mathrm{Hom}_k(V_1, J)$. Hence, (f, g) is a Shukla 2-cochain.

We next show that (f, g) is a Shukla 2-cocycle. As in Section 2 there will be four conditions to check (which in the order below correspond to associativity of multiplication, left, then right distributivity of multiplication with respect to addition, and commutassociativity of addition[3] of the extension).

1. We have

$$\begin{aligned}(\delta_b f)((x),(y),(z)) &= xf((y),(z)) - f((xy),(z)) + f((x),(yz)) - f((x),(y))z \\ &= s(x)[s(yz) - s(y)s(z)] - s(xyz) + s(xy)s(z) \\ &\quad + s(xyz) - s(x)s(yz) - [s(xy) - s(x)s(y)]s(z) = 0.\end{aligned}$$

2. If $x, y_i \in B$ and $\sum_{i=1}^{n} r_i y_i = 0$ then

$$\begin{aligned}&(\delta_b + \delta_d)(f, g)((x), (\textstyle\sum r_i(y_i))) \\ &= xg((\textstyle\sum r_i(y_i))) - g((\textstyle\sum r_i(xy_i))) + \textstyle\sum r_i f((x),(y_i)) \\ &= s(x)(\textstyle\sum r_i s(y_i)) - \textstyle\sum r_i s(xy_i) + \textstyle\sum r_i[s(xy_i) - s(x)s(y_i)] = 0.\end{aligned}$$

3. Same elements as before:

$$\begin{aligned}&(\delta_b + \delta_d)(f, g)((\textstyle\sum r_i(y_i)), (x)) \\ &= g((\textstyle\sum r_i(y_i x))) - g((\textstyle\sum r_i(y_i)))x - \textstyle\sum r_i f((y_i),(x)) \\ &= \textstyle\sum r_i s(y_i x)) - [\textstyle\sum r_i s(y_i)]s(x) - \textstyle\sum r_i[s(y_i x) - s(y_i)s(x)] = 0.\end{aligned}$$

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[3] $(x + y) + (z + w) = (x + z) + (y + w)$ for every element x, y, z, and w.

4. A k-homogeneous base elements of V_2 is given by $(\sum_{i=1}^m k_i(n_i))$, where $\sum_{i=1}^m k_i n_i = 0$ in V_0 and $n_i = \sum_{j=1}^{m_i} r_{ij}(x_{ij})$ such that $\sum_j r_{ij} x_{ij} = 0$ in B for every i. Then

$$(\delta_d g)((\sum k_i(n_i))) = -g(\sum k_i(n_i)) = -\sum_{i=1}^m k_i g(n_i)$$

$$= -\sum_i k_i g\left(\left(\sum_i r_{ij}(x_{ij})\right)\right) = -\sum_i k_i \sum_j r_{ij} s(x_{ij})$$

$$= -g\left(\left(\sum_{i,j} k_i r_{ij}(x_{ij})\right)\right) = -g(0) = 0,$$

since the last nontrivial argument is equal to $\sum k_i n_i$.

It follows from linearity that (f, g) is a Shukla 2-cocycle. By hypothesis, $HS^2(B, J) = 0$, so that there exists a normalized Shukla 1-cochain $h \in \mathrm{Hom}_k(V_0, J)$ such that $(f, g) = (\delta_b h, \delta_d h)$. Consider the map $\psi : B \to A$ defined by

$$\psi(x) = s(x) + h((x))$$

for every $x \in B$. We next show that ψ is an algebra homomorphism and a right inverse of p, finishing the proof when $J^2 = 0$ since we take $S = \psi(B)$. Towards this end note that

1. $p \circ \psi = Id_B$ since $p \circ s = Id_B$ and $p \circ h = 0$;
2. ψ is k-linear: given $\sum_{i=1}^n r_i x_i \in B$, note that

$$s(\sum r_i x_i) - \sum r_i s(x_i) = g(((\sum r_i x_i) - \sum r_i(x_i)))$$

$$= (\delta_d h)(((\sum r_i x_i) - \sum r_i(x_i))) = \sum r_i h((x_i)) - h((\sum r_i x_i));$$

whence $\psi(\sum r_i x_i) = s(\sum r_i x_i) + h((\sum r_i x_i)) = \sum r_i s(x_i) + \sum r_i h((x_i)) = \sum r_i \psi(x_i)$.

3. ψ is multiplicative: $\psi(x)\psi(y) = [s(x) + h((x))][s(y) + h((y))] = s(x)s(y) + h((x))y + xh((y)) = s(xy) - f((x),(y)) + \delta_b h((x),(y)) + h((xy)) = s(xy) + h((xy)) = \psi(xy)$.

To finish the proof we employ the well-known induction argument of Hochschild on the degree n of nilpotency of the ideal J. First, note that $0 \to J/J^2 \to A/J^2 \to B \to 0$ splits by our argument for $n = 2$. Then there exists a subalgebra C of A such that $0 \to J^2 \to C \to B \to 0$ is an exact sequence, which splits as well by the induction hypothesis. Take S to be the image of B in C under the splitting homomorphism. S satisfies $A = S \oplus J$, which completes the proof. □

Remark. Our thanks are due to the referee for noting that this result can be obtained through the route of Beck's classification of principal homogeneous objects with the triple-theoretic cohomology H^1 in his thesis [2] and an application of Barr's article [1] to show that Shukla cohomology H^2 is isomorphic with the cotriple H^1.

4. Shukla cohomology and ring theory

What has the Wedderburn principal theorem done for for algebras? Aside from its philosophical import on the algebraic community, the principal theorem seems to have led directly to a classification of algebras only in the case of algebras over an algebraically closed field with square-zero radical (see [9]). Otherwise one is left with the not small task of describing the multiplicative bimodule structure of the radical over the semisimple subalgebra.

A purely additive analog of Wedderburn's principal theorem – and one with enormous success for the classification problem within its own category – is the following proposition.

Proposition 4.1. *A divisible abelian group D with torsion subgroup D_t has a torsion-free subgroup E such that $D = E \oplus D_t$.*

Thus the classification of divisible groups is accomplished with the dimension of the rational vector space E and the $\mathbb{Z}(p^\infty)$-group components of D_t. Now any abelian group has a maximal torsion subgroup D_t; if D is divisible, then D_t is an injective $\mathbb{Z}$-module, so that $\mathrm{Ext}^1_{\mathbb{Z}}(A, D_t) = 0$ for every abelian group A. Whence the proposition above may be stated more generally in the cohomological statement below (by recalling the 1–1 correspondence of Ext^1 groups with extensions of modules [8]).

Proposition 4.2. *An abelian group D is isomorphic to $D_t \times D/D_t$ if*

$$\mathrm{Ext}^1_{\mathbb{Z}}(D/D_t, D_t) = 0.$$

As an abelian group, any ring R will have a maximal torsion subgroup R which is in fact an ideal, and behaves like a radical. R/R_t will of course be a torsion-free ring – not an idle remark since we have the following subtle relation between Shukla cohomology and the Ext groups when $n = 2$.

Proposition 4.3. (Cegarra, Garzon [3]). *Suppose k is an integral domain and A is a unital k-algebra that is torsion-free as a k-module. Then*

$$HS^2(A, M) = \mathrm{Ext}^2_{A \otimes_k A^{\mathrm{op}}}(A, M)$$

for every A-bimodule M.

This theorem requires a subtle proof but should not come as a major surprise to the reader because if we had the stronger condition of projectivity on the k-module A, then for all $n \geq 0$ we have $HS^n(A, M) \cong HH^n(A, M) \cong \mathrm{Ext}^n_{A \otimes A^{\mathrm{op}}}(A, M)$, the last identity noted by Cartan and Eilenberg. One can now state the main theorem (3.1) for torsion-free rings ($\mathbb{Z}$-algebras) or torsion-free algebras over an integral domain in the more familiar terms of Ext^2 groups.

It would indeed be nice to state a theorem as elegant as the principal theorem for a certain class of rings – that they automatically factor as a direct sum of some radical and a semisimple subring. But finding such a class of rings without just a rediscovery of finite-dimensional algebras over a perfect field, or an awkward reference to cohomological dimension of a factor ring, seems to present difficulties. The next example is a basic obstruction to several attempts at such generality. For example, it is a counterexample to the two conjectures:

1. A torsion-free left (right) artinian ring is always a direct sum of a semisimple subring and its nil radical.

2. Goldie's third theorem (a right noetherian principal left ideal ring R is a direct sum of a semiprime principal left ideal ring and an artinian ring) has cohomological proof along the lines of finding the quotient of R by its prime radical and applying Theorem 3.1.

Example. Let $A = \mathbb{Q}(t_1, t_2)$ be the field of rational functions in two indeterminates. Consider the A-valued Hochschild 2-cocycle f defined by

$$f(u,v) = \frac{\delta u}{\delta t_1}\frac{\delta v}{\delta t_2}.$$

Define multiplication on the $\mathbb{Q}$-vector space $A \oplus A$ by

$$(u,v)(w,z) = (uz + vw + f(v,z), vz)$$

and call the resulting $\mathbb{Q}$-algebra B. Since any 2-coboundary $(bg(a,a') = ag(a') - g(aa') + g(a)a')$ is clearly symmetric over the commutative algebra A, it is clear that f is noncobounded since it is not symmetric in its variables. Hence, B is a torsion-free artinian ring with no direct sum decomposition $B = A \oplus J$ as a consequence of the pairing of the Hochschild cohomology group $HH^2(A,A)$ with square-zero extensions of A by A. Also, B is a principal left ideal ring with prime radical A and quotient ring A of projective $A \otimes A^{\mathrm{op}}$-dimension 2.

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